

Lecture 1. Normalizing flows

Introduction to Bayesian Statistical Learning II

20.05.2025 Instructors: Alina Bazarova, Jose Robledo

Brief recall on the Bayesian concepts

$$\textit{posterior} = \frac{\textit{prior} \times \textit{likelihood}}{\textit{evidence}}$$

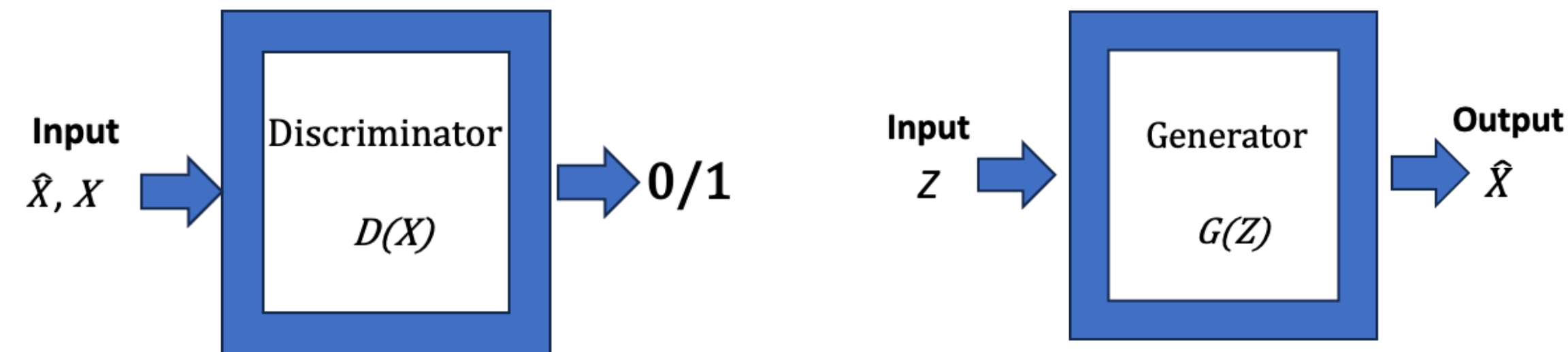
$$P(A | X) = \frac{P(A)P(X | A)}{P(X)}$$

Where A are the parameters and X is the data (discrete case)

$$p(\theta | x) = \frac{p(\theta)p(x | \theta)}{\int p(x)p(x | \theta)d\theta}$$

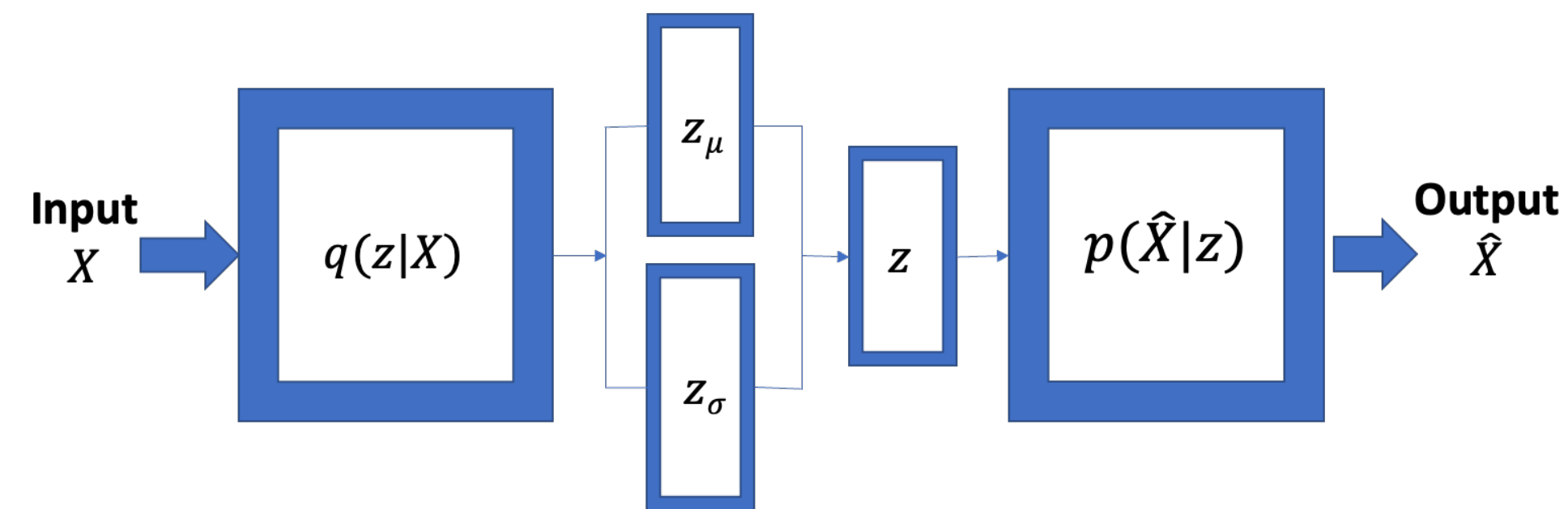
Where θ are the parameters, and $p(x)$ is a probability density function (continuous case)

Generative models



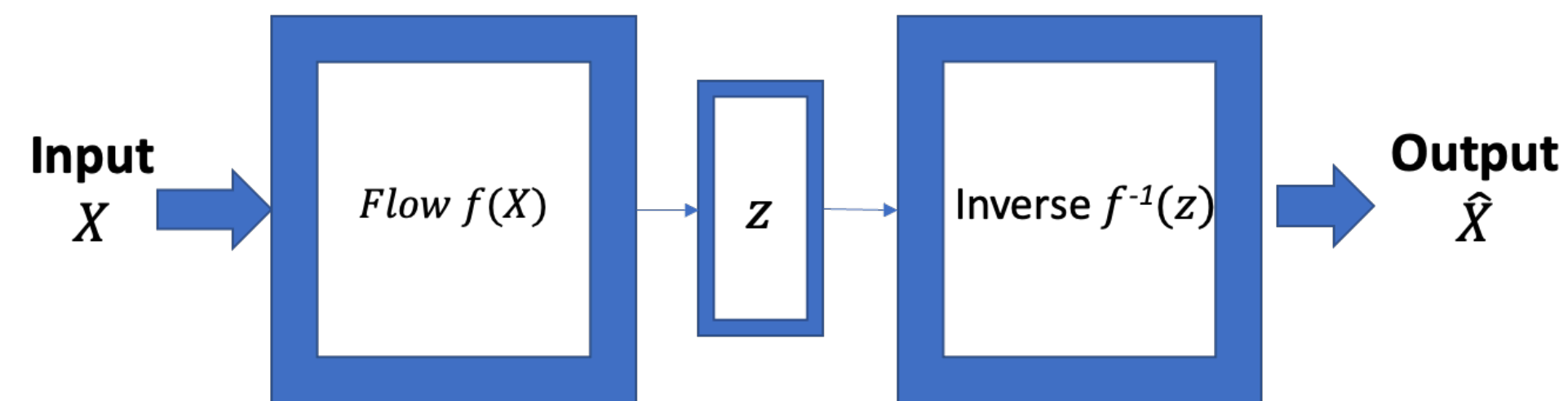
GAN: generator and discriminator trained together

No likelihood estimate



VAE: implicitly learns the distribution of the data

Latent space has a **lower** than input dimension



Normalizing flows: learns exact likelihood estimate, uses

A chain of invertible functions. Latent space has the **same** dimension as input

What do normalizing flows have to do with Bayesian inference?

- Normalizing flows are capable of learning **exact likelihood** estimate, and therefore can be a powerful tool in **approximate** Bayesian methods such as Simulation Based Inference, especially in cases when likelihood is **intractable**

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- Normalizing flows are capable of learning **exact likelihood** estimate, and therefore can be a powerful tool in **approximate Bayesian** methods such as Simulation Based Inference, especially in cases when likelihood is **intractable**
- Normalizing flows represent a series of transformations of an initial simple distribution - can be viewed as our **prior beliefs** on the posterior distribution

More concrete...

Main idea: We wish to map **simple distributions** which are easy to sample from and evaluate densities to **complex ones** (which are learned via data)

Change of variables

Let Z and X be random variables, such that $X = f(Z)$, $Z = f^{-1}(X)$,
where $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ then $p_X(x) = p_Z(f^{-1}(x)) \left| \det\left(\frac{\partial f^{-1}(x)}{\partial x}\right) \right|$ holds.

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- x and z are continuous and of the same dimension

- $\frac{\partial f^{-1}(x)}{\partial x}$ is a Jacobian $n \times n$ matrix, where each (i, j) entry is $\frac{\partial f^{-1}(x)_i}{\partial x_j}$

Normalizing flow models

Latent variables Z and observed variables X , $f_\theta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a mapping between X and Z , which is deterministic and invertible.

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Using **change of variables**, the marginal likelihood $p(x)$ is given by

$$p_X(x; \theta) = p_Z(f_\theta^{-1}(x)) \left| \det\left(\frac{\partial f_\theta^{-1}(x)}{\partial x}\right) \right|$$

Key requirements:

1. f_θ is invertible
2. x and z have the same dimension
3. Jacobian computation has to be efficient

Normalizing flow models. Examples

Planar flow

$x = f_{\theta}(z) = z + u \mathbf{h}(w^T z + b)$, where u, w, b are trainable parameters

$$\left| \det\left(\frac{\partial f_{\theta}(z)}{\partial z}\right) \right| = |1 + h'(w^T z + b)u^T w| \quad \textbf{NB: } h'(w^T z + b)u^T w \geq -1, h \text{ is invertible}$$

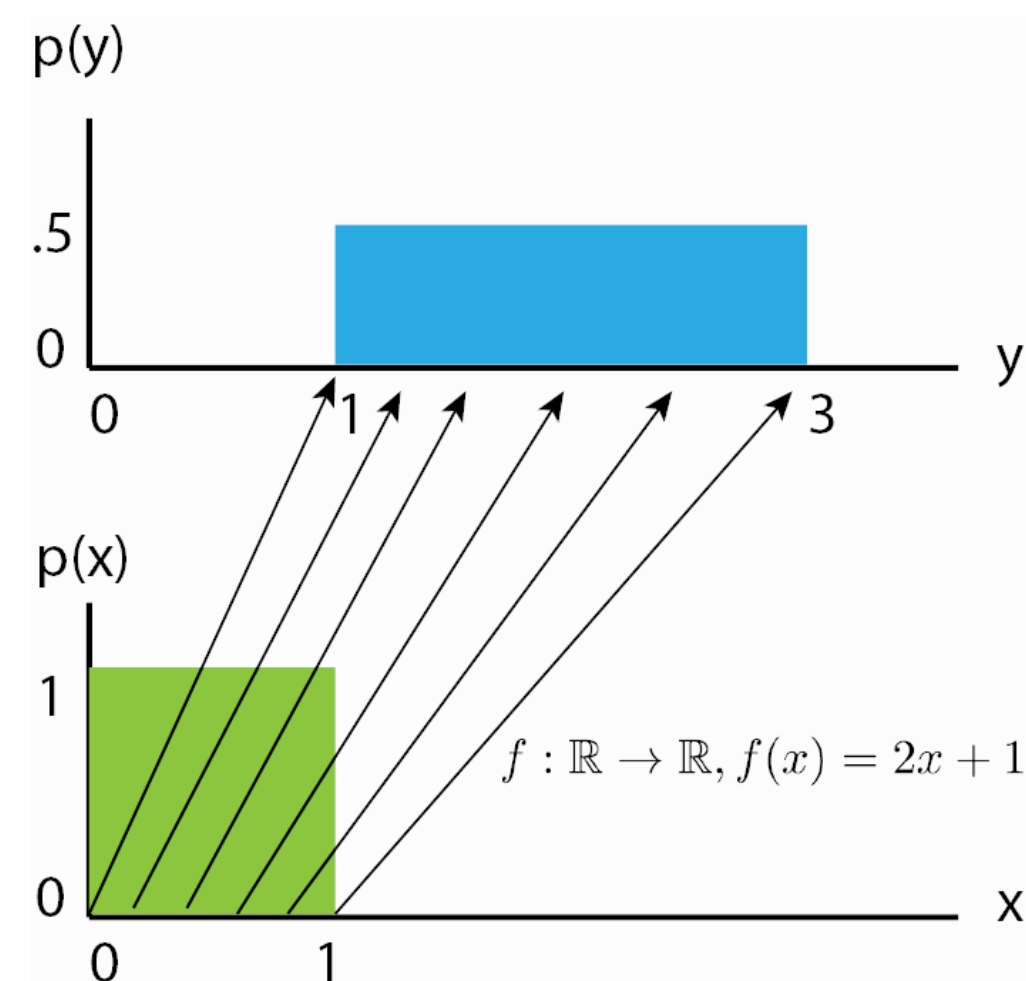


Figure credit Eric Jang <https://blog.evjang.com/2018/01/nf1.html>

<— Illustration: affine shift $p_X(x) = p_Z(f^{-1}(x)) \left| \det\left(\frac{\partial f^{-1}(x)}{\partial x}\right) \right|$

Transforming $U[0,1]$ distribution using $f(x) = 2x + 1$

NB: we are dealing with probability density functions,
hence the transformed volume has to **integrate to 1!**

Normalizing flow models. Examples

Nonlinear Independent Components Estimation (NICE)

Partitions x into two **disjoint subsets** x_1 and x_2

Forward mapping: $z_1 = x_1, z_2 = x_2 + m_\theta(x_1)$, where the first one is an **identity mapping**,

and m_θ is a **neural network**

Reverse mapping: $x_1 = z_1, x_2 = z_2 - m_\theta(x_1)$

The **Jacobian** of the forward mapping is **lower-triangular**, determinant is equal to 1 (volume preserving transform).

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Real Non-Volume Preserving (RealNVP)

$z_2 = \exp(s_\theta(x_1)) \odot x_2 + m_\theta(x_1)$ Will look closer in the jupyter notebook!

Jacobian is a product of the **scaling factors**!

Normalizing flow models. Examples

Masked Autoregressive Flow (MAF)

$$p(x) = \prod_i p(x_i | x_{1:i-1})$$

Target density is modelled as a product of one-dimensional densities, depending only on the previous values

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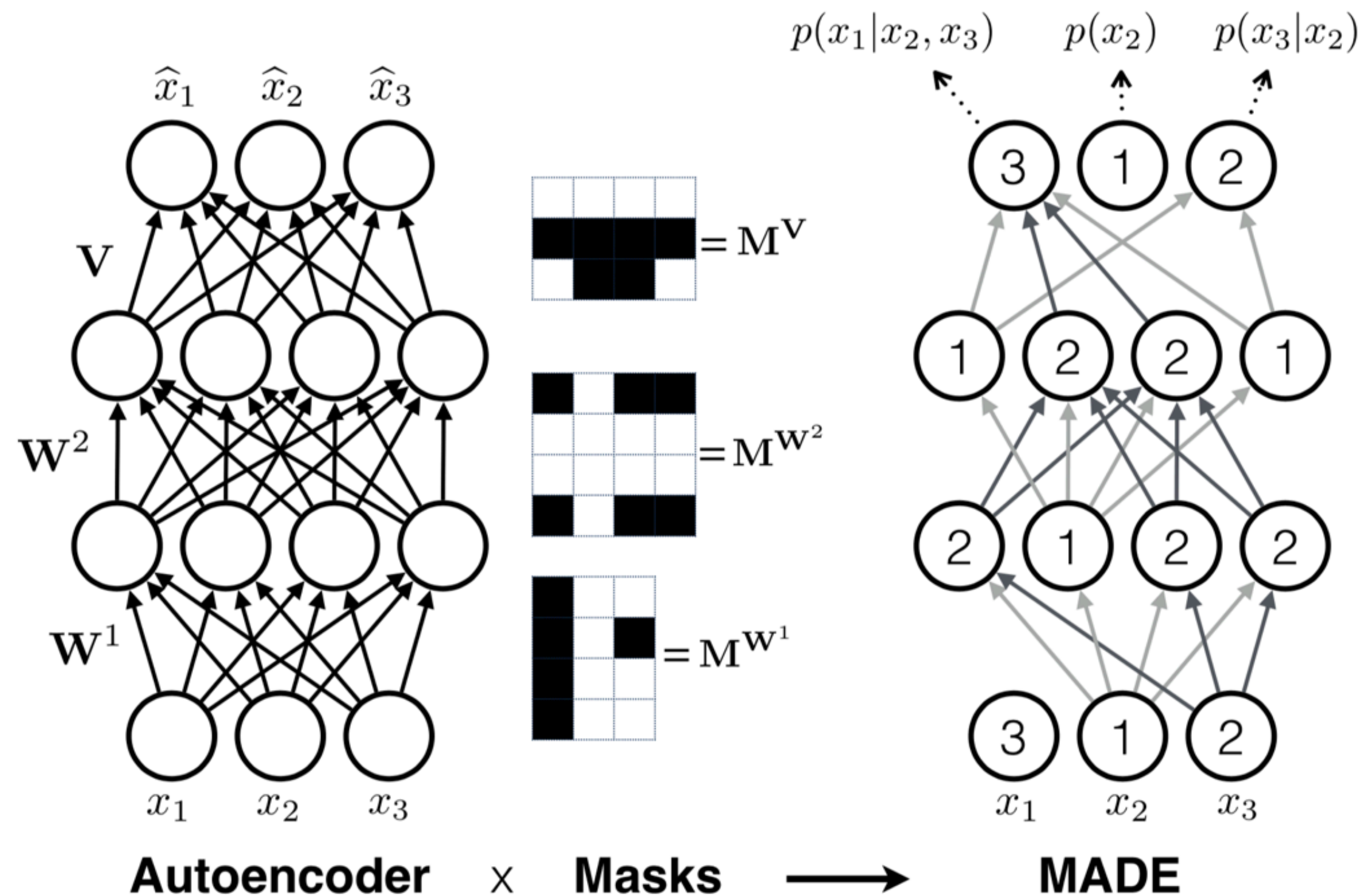
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However, once x is computed, the density estimation can be significantly **speeded up** with **MADE!**

Normalizing flow models. Examples

Masked Autoencoder for Distribution Estimation (MADE): allows to speed up MAF!

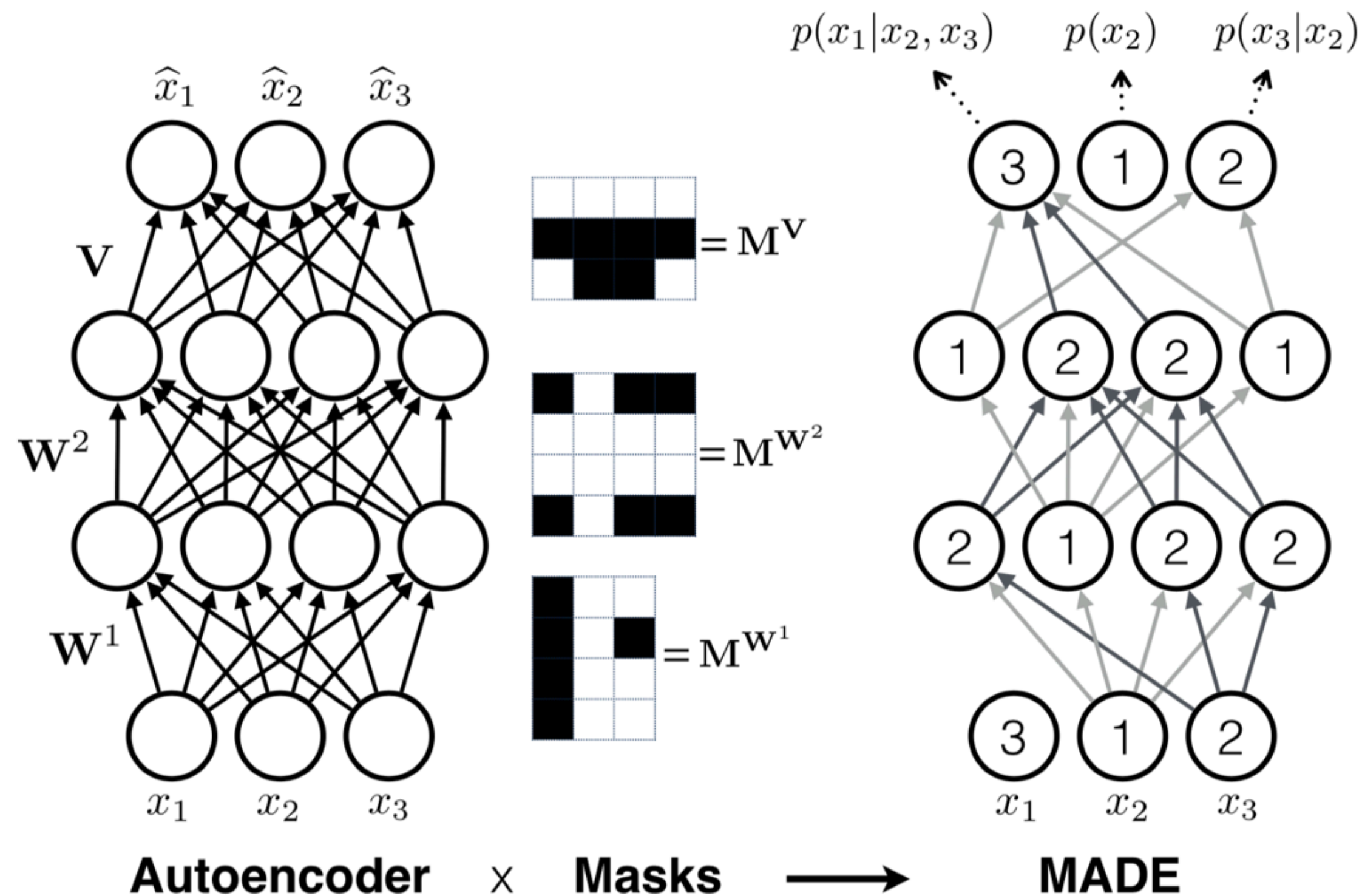
All conditional likelihoods $p(x_1), p(x_2 | x_1), \dots, p(x_D | x_{1:D-1})$ are estimated in a single pass of D threads



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MADE **masks** out connections in the network

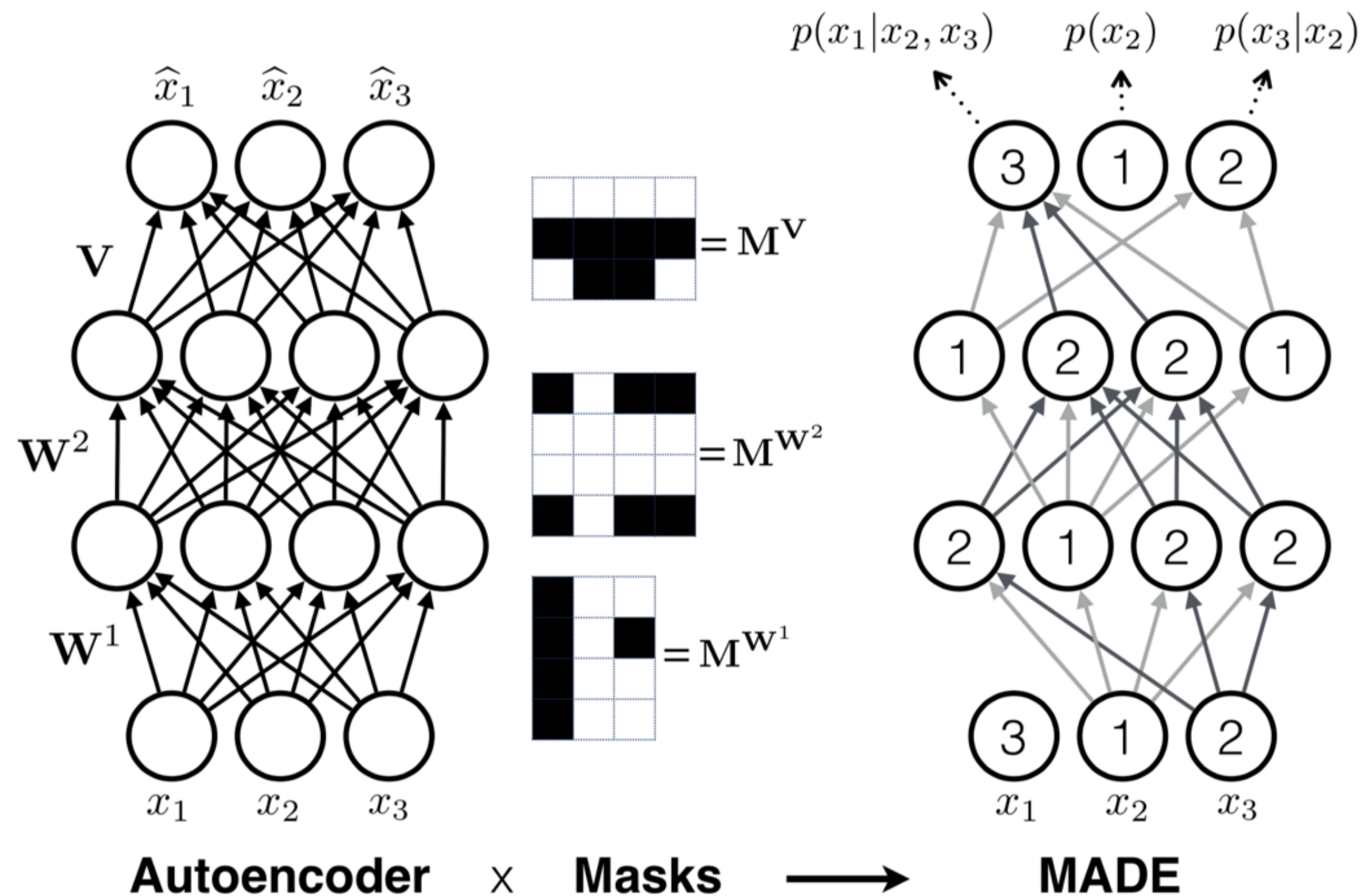
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- Order agnostic (shuffle input dimensions)
- Connectivity agnostic (resample connectivity integers)

Normalizing flow models. Examples

Inverse Autoregressive Flow (IAF)

MAF transformation $x_i = z_i \exp s_{i,\theta} + m_{i,\theta}$, $z_i \sim \mathcal{N}(0,1)$ x_i depends on $x_{1:i-1}$

Inverse $z_i = \frac{1}{\exp s_{i,\theta}} x_i - \frac{m_{i,\theta}}{\exp s_{i,\theta}}$ where $m_{i,\theta}$ and $s_{i,\theta}$ depend only on $x_{1:i-1}$

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IAF: $\tilde{x}_i = \tilde{z}_i \exp \tilde{s}_{i,\theta} + \tilde{m}_{i,\theta}$, where $\tilde{m}_{i,\theta} = \tilde{m}_{i,\theta}(\tilde{z}_{1:i-1})$, $\tilde{s}_{i,\theta} = \tilde{s}_{i,\theta}(\tilde{z}_{1:i-1})$

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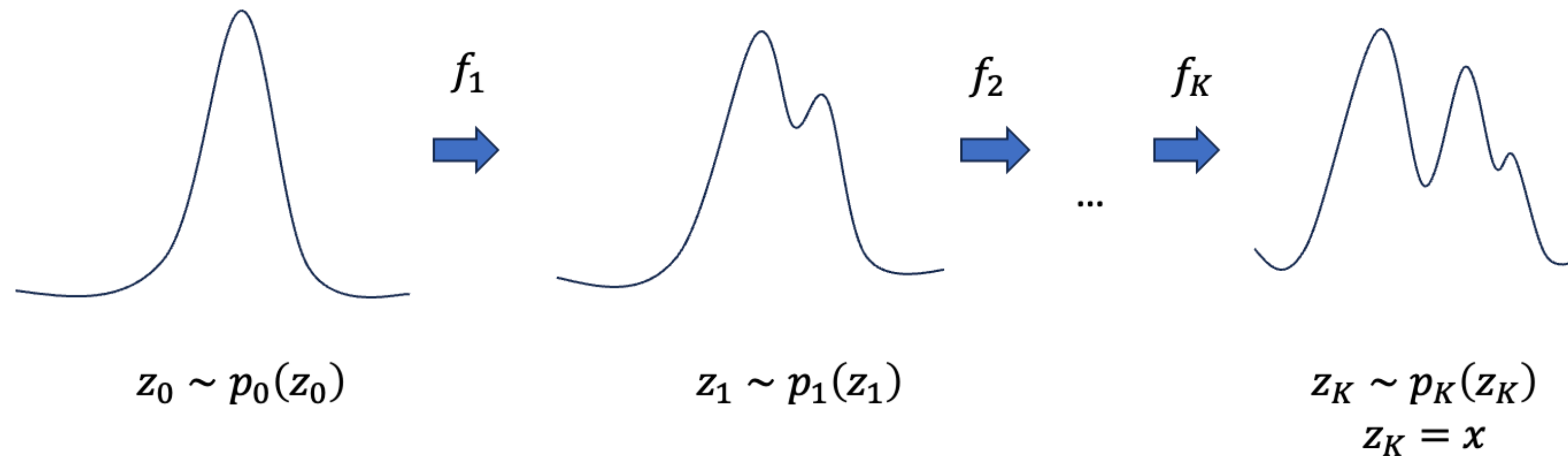
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Data generation: fast with MADE!

Density estimation: slow, as requires computation of $\tilde{z}_{1:i-1}$ since $\tilde{x}_i \sim p(\tilde{x}_i | \tilde{z}_{1:i-1})$

Training Normalising flows

In reality we apply a chain of transformations f_1, \dots, f_K to the prior density $p(z)$



Loss \leftarrow negative log-likelihood: $-\log p_z(f_1^{-1} \circ f_2^{-1} \circ \dots \circ f_K^{-1}(x)) - \sum_i \log \det \left| \frac{df_i^{-1}(z_i)}{dz_i} \right|$

With respect to function (bijector) parameters

Dequantization

- Normalizing flows operate on continuous distributions
- Most of the data we work with have discrete nature
- Hence dequantization: **converting** discrete data into **continuous** by **adding small amount of noise**

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2. Solution: **Variational** dequantization. In the above formula use learnable distribution $q_{\theta}(u | x)$, modelled via an additional normalizing flow

Jupyter notebook